Indian Institute of Information Technology Allahabad
Tentative Marking Scheme for End Sem Question Paper
Discrete Mathematical Structures

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1. Check whether the following statements are true or false. Give a proper justification. [$2 \times 6 = 12$]

   (a) Let $f : X \to Y$ be a function and let $A, B \subset Y$. Then $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

   Solution: True

   We have $x \in f^{-1}(A \cup B) \iff f(x) \in A \cup B$
   $\iff f(x) \in A$ or $f(x) \in B$
   $\iff x \in f^{-1}(A)$ or $x \in f^{-1}(B)$.

   (b) $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

   Solution: True

   Using laws of propositions, we have $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge (\neg(\neg p \wedge q)) \equiv \neg p \wedge (p \vee \neg q) \equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) \equiv F \vee (\neg p \wedge \neg q) \equiv (\neg p \wedge \neg q)$.

   (c) The set $X = \left\{ x = \sum_{i=0}^{\infty} \frac{a_i}{5^i} : a_i \in \{0, 2, 4\} \right\}$ is countable.

   Solution: False

   We know that the set $Y = \{0, 2, 4\}^{\mathbb{N}}$ is uncountable. It is obvious that the mapping $f : Y \to X$ defined as $f((x_1, x_2, x_3, \dots)) = \sum_{i=0}^{\infty} \frac{x_{i-1}}{5^i}$, is bijective.

   (d) Let G be a simple, connected graph with 13 vertices and 76 edges. Then G is Hamiltonian as well as Eulerian.

   Solution: False

   Note that it has two less vertices in comparision of number of edges in K_{13} that is $\binom{13}{2} = 78$. This means that the minimum possible degree of a vertex in G is 10 as at most two edges from one vertex in K_{13} could be removed to obtained G.

   As the minimum degree of any vertex in G is 10 then by Dirac's Theorem

G is Hamiltonian.

The graph is not necessarily Eulerain as G can be obtained from K_{13} by deleting two edges in such a way that G has four vertices of degree 11.

(e) The group of symmetries of an equilateral triangle D_3 is abelian.

Solution: False

Here $D_3 = \{1, r, r^2, s, sr, sr^2\}$. For two elements sr and sr^2 of D_3, we get $(sr)(sr^2) \neq (sr^2)(sr)$.

(f) Every abelian group is cyclic.

Solution: False

Consider an abelian group $G = \{e, a, b, ab\}$ with $a^2 = e, b^2 = e$, it is easy to see that G is not cyclic.

2. Let us define sets A and B as follows: [10]

$$A = \{g, o, l, e\} \quad \text{and} \quad B = \{\text{Your enrolment number}\}$$

For example: if your enrolment number is $iit2023175$, then $B = \{i, t, 0, 1, 2, 3, 5, 7\}$. Now, define $C = A \cup B$ and $m = |A|,\ n = |C|$, where $|A|$ denotes the number of elements in A.

We have $A = \{g, o, l, e\}$ and $C = \{e, g, i, l, o, t, 0, 1, 2, 3, 5, 7\}$.
So, $|A| = m = 4$ and $|C| = n = 12$.

(a) Let F be a finite field with $t_* = 5^n$ elements. Prove that $x^{t_*} = x$ for all nonzero $x \in F$.

Solution: We know that if F is field then $(F \setminus \{0\}, .)$ forms a group of order $t_* - 1$. Take $|F| = 5^{12}$, then $x^{5^{12}-1} = 1\ \forall\ x \in\ F \setminus \{0\}$ by using that order of an element divide order of the group. Hence, $x^{5^{12}} = x\ \forall\ x \in F$. [2]

(b) Construct a non-identity group homomorphism $\Phi : \mathbb{Z}_n \to \mathbb{Z}_n$, where $\mathbb{Z}_n$ is the group of integers modulo n.

Solution: If $\Phi : \mathbb{Z}_n \to \mathbb{Z}_n$ is a group homomorphism. Then, for $x \in \mathbb{Z}_n$, $f(x) = f(1 + 1 + \cdots + 1)(x\text{-times}) = xf(1)$. [1]

Take $f(1) = a \neq 1 \in \mathbb{Z}_n$. [1/2]

So $f(x) = ax$, for some $a \in \mathbb{Z}_n$ is a non-identity group homomorphism.[1/2]

(c) Construct (if possible) a self-complimentary graph with n vertices; otherwise, give a reason.

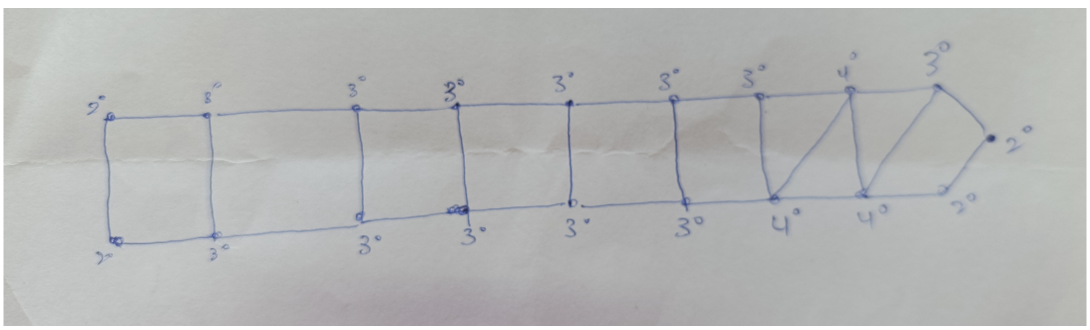
Solution: For any graph G with n vertices, $|E(G)| + |E(\bar{G})| = \frac{n(n-1)}{2}$. Since G is self-complimentary graph. $|E(G)| = |E(\bar{G})|$. This implies that $|E(G)| = \frac{n(n-1)}{4}$. Here, $n = 12$ so such graph exists. [3]

(d) Construct a simple graph with 28 edges wherein m vertices of degree 2, $m-1$ vertices of degree 4 and remaining of degree 3.

Solution: For any graph G,

$$\sum_{v \in V} v = 2|E|.$$

Suppose that total number of vertices is x. Then $4.2+3.4+(x-7).3 = 2.28$. On solving this, we get $x = 19$. [2]



[1]

3. Suppose you are given a Cadbury Dairy Milk Chocolate bar with 10 pieces. In how many ways, you can distribute the 10 pieces (considering all possible cases, including not giving any piece to some of them) to your 5 friends? [3]

Solution: Suppose each friend is given x_i number of pieces.

$$x_1 + x_2 + x_3 + x_4 + x_5 = 10, \quad 0 \leq x_i \leq 10, \quad \forall\ i = 1, 2, 3, 4, 5. \qquad [1]$$

The coefficient of x^{10} in the expansion of $(1 + x + x^2 + x^3 + \ldots + x^{10})^5$ is total number of ways that 10 pieces given to 5 friends.

$$\begin{aligned}
&(1 + x + x^2 + x^3 + \ldots + x^{10})^5 \\
&= (1 - x^{11})^5 (1 - x)^{-5} \\
&= (1 - x^{11})^5 \sum_{k=0}^{\infty} \binom{-5}{k} (-1)^k x^k \\
&= (1 + 15x^{44} + 10x^{22} - 5x^{11} - 10x^{33} - x^{55}) \sum_{k=0}^{\infty} \binom{-5}{k} (-1)^k x^k
\end{aligned}$$

So, the coefficient of x^{10} is $(-1)^{10}\binom{-5}{10}$. [2]
Note that: $\binom{-n}{r} = \frac{-n(-n-1)...(-n-r+1)}{r!}$.
Alternative solution: the number of r-combinations from a set with n elements when repetition of elements is allowed is $\binom{n+r-1}{r}$ [1]
here for given problem $n = 5$ and $r = 10$.
$\binom{5+10-1}{10} = \binom{14}{10} = 1001$. [2]

4. (a) Prove that any simple connected planar graph must have a vertex of degree 5 or less. [3]

   Solution: For a simple connected planar graph $e \leq 3v - 6$ (i)
   Suppose this has no vertex of degree 5 or less. i.e. $deg(v_i) \geq 6$ for each vertex v_i. By Handshaking lemma, $\sum_{v_i} deg(v_i) = 2e \implies 6v \leq 2e$, a contradiction (because from $(i), 2e < 6v$).

   (b) Show that a simple graph G with n vertices is connected if it has more than $(n-1)(n-2)/2$ edges. [3]

   Solution: Let G be a simple graph which has more than $(n-1)(n-2)/2$ edges. Suppose G is not connected. Then G has two subgraphs (say G_1 and G_2). Let G_1 has k-vertices then G_2 has $n-k$ vertices. So,
   $|E(G)| \leq \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} = \frac{1}{2}(2k^2 + n^2 - 2nk - n)$ or
   $|E(G)| \leq \frac{1}{2}(n-1)(n-2) + (k-1)(k-n+1)$. $(*)$
   Since $k \geq 1$ so $k - 1 \geq 0$ and $k \leq n - 1$ so $(k - n + 1) \leq 0$. Thus, by $(*), |E(G)| \leq \frac{1}{2}(n-1)(n-2)$, a contradiction. Therefore, G is connected.

5. Find all solutions of the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3} + 7^n$, with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$. [5]

Solution:

The characteristic equation of the associated homogeneous recurrence relation is given as

$$r^3 - 6r^2 + 11r - 6 = 0. \qquad [1]$$

The characteristic root is $r = 1, 2, 3$.

The solution of the recurrence relation are of the form

$$a_n^{(h)} = \alpha_1 1^n + \alpha_2 2^n + \alpha_3 3^n. \qquad [1]$$

Further, 7 is not the root of associated linear homogeneous equation. Therefore, $a_n^{(p)} = \beta 7^n$. [1]

Also, $a_n^{(p)}$ will satisfy the given recurrence relation, that is,

$$\beta 7^n = 6\beta 7^{n-1} - 11\beta 7^{n-2} + 6\beta 7^{n-3} + 7^n$$
$$\beta = \frac{6\beta}{7} - \frac{11\beta}{7^2} + \frac{6\beta}{7^3} + 1.$$

This gives $\beta = \frac{343}{120}$. Consequently, the solution is

$$a_n = a_n^{(h)} + a_n^{(p)} = \alpha_1 1^n + \alpha_2 2^n + \alpha_3 3^n + \frac{343}{120}7^n. \qquad [1]$$

Now, using the initial conditions, we get [1]

$$a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3 + \frac{343}{120}$$
$$a_1 = 5 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \frac{2401}{120}$$
$$a_2 = 15 = \alpha_1 + 4\alpha_2 + 9\alpha_3 + \frac{16807}{120}.$$

Solving these equations, we get $\alpha_1 = \frac{-23}{24}, \alpha_2 = \frac{338}{5}, \alpha_3 = \frac{-327}{8}$.

6. What is the generating function for $\{a_k\}$, where a_k is the number of solutions of

$$x_1 + x_2 + x_3 + x_4 = k,$$

when x_1, x_2, x_3, and x_4 are integers with $x_1 \geq 3,\ 1 \leq x_2 \leq 5,\ 0 \leq x_3 \leq 4$, and $x_4 \geq 1$? [4]

Solution:

The generating function is given as

$$\begin{aligned}
\phi(x) &= \Big((x^3 + x^4 + x^5 + \ldots)(x + x^2 + x^3 + x^4 + x^5) \\
&\qquad (1 + x + x^2 + x^3 + x^4)(x + x^2 + x^3 + \ldots)\Big) && [1] \\
&= x^5(1 + x + x^2 + x^3 + x^4)^2(1 + x^1 + x^2 + x^3 + \ldots)^2 \\
&= x^5\Big(\frac{1 - x^5}{1 - x}\Big)^2\Big(\frac{1}{1 - x}\Big)^2 && [1] \\
&= x^5(1 - x^5)^2(1 - x)^{-4} \\
&= x^5(1 + x^{10} - 2x^5)\sum_{k=0}^{\infty}\binom{-4}{k}(-1)^k x^k. && [2]
\end{aligned}$$